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AN EXACT SOLUTION OF A LINEARIZED PROBLEM OF THE RADIATION OF MONOCHROMATIC INTERNAL WAVES IN A VISCOUS FLUID†

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The eigenvalue method is used to construct an exact solution of the linearized boundary-value problem of the generation of internal waves in an exponentially stratified fluid, when the source is part of a plan which vibrates along its surface. The spatial structure of the solution obtained describes two well-known types of wave beams—unimodal and bimodal. In the limiting cases the phase pattern of the waves is identical with well-known asymptotic forms and laboratory experiments. The exact solution is compared with the solution of the model problem of the generation of waves by force sources, constructed using homogeneous fluid theory. The phase patterns of the waves in both cases agree everywhere with the exception of critical angles, when the wave propagates along the radiating surface. The amplitudes of the radiated waves are the same only for certain ratios of the angles of inclination of the plane and the direction of propagation of the beams. © 1999 Elsevier Science Ltd. All rights reserved.

Problems of the generation of motions by a periodic source in a homogeneous and stratified ideal fluid [1] or viscous fluid [2, 3] are traditionally investigated experimentally and theoretically [4, 5]. For the equations of internal waves in a stratified fluid it has not been possible up to now to obtain the form of the moving body for which the problem of the generation of motions has an exact solution even in the linear formulation. One of the widely used approaches consists of replacing the actual body by force sources [6] or mass sources [7], the characteristics of which are adopted from the theory of a homogeneous fluid. The structure of the wave beam, calculated using these models, agrees qualitatively with observations at large distances from the source. The absolute values of the displacements in this approach turn out to be too low and are corrected using empirical coefficients [8].

Detailed laboratory observations show that there is an intermediate region between the wave beam formed and the vibrating body, but the motions in this region are not purely wave motions [7] and include both flow of the boundary-layer type [9], and more complex vortex motions, which are not described by universal models.

The purpose of this paper is to consider a special problem of the hydrodynamics of a vibrating body, which admits of an exact solution both in the case of a one-dimensional and exponentially stratified viscous fluid, when internal waves are radiated into the medium. A consideration of the viscous boundary flows on the radiating surface is a fundamental feature. The form of the radiator and the value of the displacement amplitude are chosen from the condition which enables non-linear effects to be neglected. By analysing the solutions obtained the conclusion is drawn that replacing the vibrating body by a set of force sources is inadequate and a new algorithm for solving the problem of the radiation of internal waves is proposed.

1. THE EQUATIONS OF MOTION, BOUNDARY CONDITIONS AND THE MODEL OF THE SOURCE OF MOTIONS

Suppose x and z are horizontal and vertical coordinates, $\rho_0(z)$ is the unperturbed density distribution, (u_x, u_z) are the components of the fluid velocity, ν is the kinematic viscosity, g is the acceleration due to gravity, directed opposite to the z axis, and (f_x, f_z) is the distribution of the force sources, which excite internal waves. The effects of diffusion of the stratifying component are ignored. We will introduce a stream function Ψ such that $u_x = -\partial\Psi/\partial z$, $u_z = \partial\Psi/\partial x$. Eliminating from the system of linearized hydrodynamic equations, describing the radiation and propagation of two-dimensional internal waves [10], excited by the force sources, all variables apart from Ψ, we obtain, in the Boussinesq approximation

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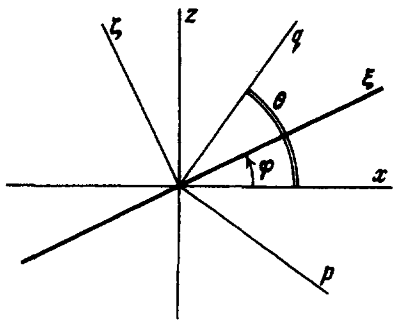

Fig. 1.

$$\left[-\frac{\partial^2}{\partial t^2}\Delta + \nu\frac{\partial}{\partial t}\Delta^2 - N^2(z)\frac{\partial^2}{\partial x^2}\right]\Psi = \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)$$
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad N^2(z) = -g\frac{d\ln\rho_0(z)}{dz} \tag{1.1}$$

Henceforth we will use several systems of coordinates (they are shown in Fig. 1): the natural system (x, z), in which the z axis is directed opposite to the gravity force, a local system (ξ, ζ), connected with the radiating surface, which makes an angle φ with the horizontal (the ξ axis is in the plane and the ζ axis is normal to it), and a co-moving system of coordinates (p, q), connected with the radiated beam, which propagates at an angle θ to the horizontal (the q axis is directed along the beam and the p axis is perpendicular to it), the relation between which is given by

$$x = \xi\cos\varphi - \zeta\sin\varphi = p\sin\theta + q\cos\theta, \tag{1.2}$$
$$z = \xi\sin\varphi + \zeta\cos\varphi = -p\cos\theta + q\sin\theta$$

Everywhere below it is assumed that the source motion is monochromatic, so that $\upsilon(\xi, t) = \upsilon_0(\xi)e^{-\omega t}$, and the common factor $e^{-i\omega t}$ is henceforth omitted. Moreover, we will assume that the fluid is stratified exponentially and the buoyancy frequency $N > \omega$ is independent of z. Taking this into account and using transformation (1.2), we obtain the following equation for the internal waves in (ξ, ζ), coordinates

$$\left[\omega^2\Delta - N^2\left(\cos\varphi\frac{\partial}{\partial\xi} - \sin\varphi\frac{\partial}{\partial\zeta}\right)^2 - i\omega\nu\Delta^2\right]\Psi(\xi,\zeta) = -\frac{i\omega}{\rho_0}\left(\frac{\partial f_\xi}{\partial\zeta} - \frac{\partial f_\zeta}{\partial\xi}\right)$$
$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2} \tag{1.3}$$

The velocity components are expressed in terms of the stream function by the relations

$$u_\xi = -\partial\Psi/\partial\zeta, \quad u_\zeta = \partial\Psi/\partial\xi \tag{1.4}$$

The source of perturbations is an infinite plane, which makes an arbitrary angle φ with the horizontal (Fig. 1), and which possesses anisotropic mechanical properties: it is infinitely stiff in a transverse direction and extensible and compressible in a longitudinal direction. The whole plane is at rest with the exception of a certain part of it, which vibrates in a longitudinal direction in a specified manner $\upsilon(\xi, t)$, where t is the time. The boundary conditions for the velocity field u_ξ, u_ζ, excited by the plane, which vibrates with a constant frequency ω, consist of the no-slip and impermeability conditions for points of the plane and the decay condition for all the motions at infinity and, taking relations (1.4) into account, have the form

$$\partial\Psi/\partial\zeta|_{\zeta=\pm 0} = -\upsilon_0(\xi), \quad \Psi|_{\zeta=\pm 0} = 0, \quad \Psi|_{\zeta=\pm\infty} = 0 \tag{1.5}$$

Equation (1.3) is then solved for the stratified medium with boundary conditions (1.5) when there are no force sources ($f_\xi = f_\zeta = 0$). In the limiting case when $N > 0$, the solution obtained describes motions excited by a plane which vibrates in a uniform fluid. We then solve the inhomogeneous equation (1.3) when there are forces in a stratified fluid, whence by taking the limit as $N \to 0$ we obtain a solution with force sources in a homogeneous fluid. A comparison of the solutions of the first and second problems for a homogeneous fluid enables us to obtain the distribution of the force sources, which is substituted into the solution of the second problem for a stratified fluid. Finally, we compare the exact solution for a stratified fluid with the results of calculations obtained when force sources are present.

2. PERIODIC MOTION OF A PLANE IN A STRATIFIED FLUID

When describing the motions excited in a stratified fluid by a vibrating plane, the solution of the homogeneous equation corresponding to (1.3) with boundary conditions (1.5) will be sought in the form of an expansion in plane waves

$$\Psi = \int_{-\infty}^{+\infty} [B_j(k)e^{i\varkappa_j\zeta} + B_{j+1}(k)e^{i\varkappa_{j+1}\zeta}]e^{ik\xi}dk, \quad j = \begin{cases} 1, & \zeta > 0 \\ 3, & \zeta < 0 \end{cases} \tag{2.1}$$

The wave numbers $\varkappa_j(k)$ are the solutions of the dispersion equation of the internal waves which, in the local system of coordinates (ξ, ζ) (Fig. 1), has the form

$$\omega^2(\varkappa^2 + k^2) - N^2(\varkappa \sin\varphi - k\cos\varphi)^2 + i\omega\nu(\varkappa^2 + k^2)^2 = 0 \tag{2.2}$$

where the indexation of the roots is chosen so that $\operatorname{Im}\varkappa_1(k) > 0$, $\operatorname{Im}\varkappa_2(k) > 0$, while $\operatorname{Im}\varkappa_3(k) < 0$. It was shown in [11] that the roots of Eq. (2.2) can be split into two pairs with opposite signs of the imaginary parts for all k. The presence of non-zero imaginary parts in $\varkappa_j(k)$ ensures that the integrals in (2.1) converge and also ensure that the solutions obtained are analytic in the whole space.

In the case of low viscosity approximate expressions were obtained for $\varkappa_j$ [11, formula (2.1)]. Travelling internal waves [3] correspond to the roots $\varkappa_1$ and $\varkappa_3$, while boundary flows correspond to the roots $\varkappa_2$ and $\varkappa_4$. These motions are characterized by natural scales of spatial variability [11]

$$l_\nu = (\nu\sin\theta / N)^{1/2}, \quad \lambda_b = [2\nu\sin\theta/(N\,|\sin^2\theta - \sin^2\varphi|)]^{1/2} \tag{2.3}$$

where $\theta = \arcsin(\omega/N)$ is the angle which the centre lines of the beams of internal waves make with the horizontal. These asymptotic solutions diverge at the critical angles $\varphi = \pm\theta$, when the radiated wave propagates along the plane. It was shown in [11] that the exact solutions of the dispersion equation also remain finite in the neighbourhood of the critical angles.

Substituting (2.1) into boundary conditions (1.5) we obtain a system of equations in B_j. The solutions of this system

$$B_1(k) = -B_2(k) = \frac{iV(k)}{\varkappa_1 - \varkappa_2}, \quad B_3(k) = -B_4(k) = \frac{iV(k)}{\varkappa_3 - \varkappa_4}$$
$$V(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v_0(\xi)e^{-ik\xi}d\xi \tag{2.4}$$

enable us to write the exact solution of the problem of the generation of motions in a stratified viscous fluid by a vibrating plane. The eigenvalue functions and internal waves, and also the boundary flows, are identical, which confirms the comparability of these forms of motion at least in the immediate vicinity of the body itself and the need to take into account the contribution of boundary flows and energy losses when the body vibrates in an inhomogeneous medium.

The solutions for a homogeneous fluid are obtained by taking the limit as $N \to 0$. They are described by formulae (2.1) and (2.4), in which we must make the replacement

$$\varkappa_1 \to k_1, \quad \varkappa_2 \to k_2, \quad \varkappa_3 \to -k_1, \quad \varkappa_4 \to -k_2 \tag{2.5}$$

where

$$k_1(k) = i\,|k|,\quad k_2(k) = K^-(k) + iK^+(k)$$
$$K^{\pm}(k) = \left\{[(\omega^2 + \nu^2 k^4)^{1/2} \pm \nu k^2]/2\nu\right\}^{1/2} \tag{2.6}$$

If the plane vibrates as a whole, when υ_0 does not depend on ξ, from (2.1) we have $V(k) = \upsilon_0\delta(k)$, where δ is the delta-function. Substituting this relation into (2.1) we obtain expressions for the velocity components of a rapidly decaying spatially periodic flow in the case of a homogeneous fluid, which, apart from the notation, are identical with the classical Stokes and Rayleigh solutions [2].

A fairly typical situation is when the fluid motions are excited by a thin vibrating plate of width a, to which corresponds the velocity distribution

$$\upsilon_0(\xi) = \begin{cases} \upsilon_0, & |\xi| < a/2 \\ 0, & |\xi| > a/2 \end{cases}$$

the spectral density of which

$$V(k) = \frac{\upsilon_0}{\pi k}\sin\frac{ka}{2} \tag{2.7}$$

By substituting (2.7) into (2.4) we can obtain the stream function in the form of integrals which can be evaluated numerically. If the plate width $a \to \infty$ then, as can be seen from (2.7), $V(k) \to \upsilon_0\delta(k)$, and the case considered above of a plane vibrating as a whole is obtained.

In a stratified fluid the internal waves from a localized source propagate in the form of four beams, which make an angle θ with the horizontal. Further to fix our ideas, we will consider one beam, travelling to the right and upwards, with which the co-moving system of coordinates (p, q) shown in Fig. 1, is connected. Changing to this system of coordinates, using the asymptotic expressions for the roots $\varkappa_j$ [11], and also relations (2.1) and (2.4), we obtain the following expression for the wave field in the beam

$$\Psi_w = -\frac{(1+i\mu)L_\nu}{\sqrt{2}}\left|\frac{\sin(\theta-\varphi)}{\sin(\theta+\varphi)}\right|^{1/2}\int_0^\infty V[k\sin(\theta-\varphi)]\exp\left[ikp - \frac{\nu k^3 q}{2N\cos\theta}\right]dk \tag{2.8}$$
$$\mu = \operatorname{sign}(\sin^2\theta - \sin^2\varphi)$$

which holds when $\theta - \pi < \varphi < \theta$. In a similar way we obtain for the boundary flow

$$\Psi_b = -\frac{(1+i\mu)}{2}\lambda_b \upsilon_0(\xi)\exp\left(-\frac{i\mu\zeta}{\lambda_b} - \frac{\zeta}{\lambda_b}\right) \tag{2.9}$$

The spatial scale of this flow λ_b is defined by (2.3).

As can be seen from (2.9), the motions in a spatially vibrating boundary flow decrease exponentially as ζ increases, whereas their longitudinal structure repeats the dependence $\upsilon_0(\xi)$. The solutions obtained do not hold for all angles φ since the wave numbers $\varkappa_j$ have singularities when $\sin\varphi = \pm\sin\theta$.

The cases $\varphi = 0 - \pi$ and $\varphi = \theta$, when the beam in question propagates along the plane and is not free, will not be considered further. When $\varphi = -\theta$ the beam detaches itself from the plane and propagates freely. Substituting the expressions for the wave numbers $\varkappa_j$ corresponding to this case [11, formula (2.7)] into relations (2.1) and (2.4), we obtain

$$\Psi_w = -\left(\frac{\nu\sin^2 2\theta}{2N\cos\theta}\right)^{1/3}\int_0^\infty \frac{V(k\sin 2\theta)}{k^{1/3}}\exp\left[ikp - \frac{\nu k^3 q}{2N\cos\theta}\right]dk \tag{2.10}$$

The wave field of the departing beam remains finite.

Solution (2.8), (2.9) describes both the field of the internal waves, the structure of which is identical with the well-known solutions obtained in [3, 5], and the boundary flows. The effects of the removable singularity are observed not only in the generation but also in the reflection of internal waves from a rigid plane, when the boundary flows make a considerable contribution to the energy of the process [12].

3. FORCE SOURCES IN A STRATIFIED FLUID

Since in this problem the interaction of the vibrating plane and the fluid only occurs due to the components of the viscous friction forces, parallel to the plane, the structure of the force sources can be specified in the form

$$f_\xi = F(\xi)\delta(\zeta), \quad f_\zeta = 0 \tag{3.1}$$

where $F(\xi)$ is the distribution of the force sources along the plane.

We will represent the solution of the equation for the stream function (1.3), with right-hand described taking relations (3.1) into account, in the form

$$\Psi = -\frac{i\omega}{\rho_0}\int_{-\infty}^{+\infty} F(\xi')\frac{\partial G(\xi-\xi',\zeta)}{\partial\zeta}d\xi' \tag{3.2}$$

We will seek Green's function $G(\xi, \zeta)$ of the equation in the form of an expansion in plane waves

$$\begin{aligned} G = \vartheta(\zeta)\int_{-\infty}^{+\infty}[G_1(k)e^{i\varkappa_1\zeta} + G_2(k)e^{i\varkappa_2\zeta}]e^{ik\xi}dk + \\ +\vartheta(-\zeta)\int_{-\infty}^{+\infty}[G_3(k)e^{i\varkappa_3\zeta} + G_4(k)e^{i\varkappa_4\zeta}]e^{ik\xi}dk \end{aligned} \tag{3.3}$$

where ϑ is the Heaviside unit function. Here the wave numbers $\varkappa_j(k)$ are the solutions of the dispersion equation of internal waves (2.2). Substituting (3.3) into Eq. (1.3) with right-hand side $\delta(\xi)\delta(\zeta)$, we obtain a system of equations in $G_j(k)$, solving which we have

$$G_j(k) = \frac{\operatorname{sign}(j-5/2)}{2\pi\omega\nu}\prod_{n=1}^{4}{}'\frac{1}{\varkappa_j - \varkappa_n} \tag{3.4}$$

where the prime denotes that the term with $n = j$ has been dropped from the product.

As a result, the expression for the stream function has the form

$$\begin{aligned} \Psi = \frac{2\pi\omega}{\rho_0}\Bigg[\vartheta(\zeta)\int_{-\infty}^{+\infty}\mathcal{F}(k)(\varkappa_1G_1e^{i\varkappa_1\zeta} + \varkappa_2G_2e^{i\varkappa_2\zeta})e^{ik\xi}dk - \\ -\vartheta(-\zeta)\int_{-\infty}^{+\infty}\mathcal{F}(k)(\varkappa_3G_3e^{i\varkappa_3\zeta} + \varkappa_4G_4e^{i\varkappa_4\zeta})e^{ik\xi}dk\Bigg] \\ \mathcal{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\xi)e^{-ik\xi}d\xi \end{aligned} \tag{3.5}$$

where $\mathcal{F}(k)$ is the spectral density of the force sources.

The case of a homogeneous fluid is then obtained by taking the limit as $N \to 0$ and making the replacement (2.5).

Comparing solutions (2.1), (2.4) and (3.5), (3.4) we obtain that the distribution of the force sources with density

$$\mathcal{F}(k) = -2i\rho_0\nu(k_1 + k_2)V(k) \tag{3.6}$$

where $V(k)$ is the spectral representation of the velocity distribution of the plane (the last relation in (2.4)), gives the correct solution of the problem on the vibrations of a plane in a homogeneous fluid.

Using the distribution of the force sources (3.6) in the case of a stratified fluid, we obtain from (3.5) and (3.4) (changing to the co-moving system of coordinates (p, q)) the following expressions for the field of the wave beam

$$\Psi_w = \frac{(1+i)L_\nu}{\sqrt{2}}\frac{\cos(\theta-\varphi)}{\cos\theta}\int_0^\infty V[k\sin(\theta-\varphi)]\exp\left[ikp - \frac{\nu k^3 q}{2N\cos\theta}\right]dk \tag{3.7}$$

and of the boundary flow

$$\Psi_b = -\frac{(1+i)\sqrt{2}\sin\theta}{\sin^2\theta - \sin^2\varphi} L_\nu v_0(\xi)\exp\left(-\frac{i\mu\zeta}{\lambda_b} - \frac{\zeta}{\lambda_b}\right) \tag{3.8}$$

These expressions hold when $\varphi \neq -\theta$. However, the wave beam is also described by expression (3.7) when $\varphi = -\theta$.

It follows from a comparison of the corresponding formulae that, in the non-degenerate case $\varphi \neq \pm\theta$ the spatial structure of the wave beam and the boundary flow, described by the exact solution (2.8), (2.9), is identical with the solution for force sources (3.7), (3.8).

The amplitudes of the corresponding motions for these models differ considerably. To compare them we introduce the functions $\beta_w(\varphi)$ and $\beta_b(\varphi)$, which are equal to the ratio of the amplitudes of the wave beams and the boundary flows of solutions (3.7), (3.8) and (2.8), (2.9)

$$\beta_w(\varphi) = \frac{\cos(\theta-\varphi)}{\cos\theta}\left|\frac{\sin(\theta+\varphi)}{\sin(\theta-\varphi)}\right|^{1/2}, \quad \beta_b(\varphi) = \frac{2\sin\theta}{|\sin^2\theta - \sin^2\varphi|^{1/2}} \tag{3.9}$$

In Fig. 2 we present calculations of the ratios of the amplitudes of the boundary flows $\beta_b(\varphi)$ (curves a and b) and of the internal waves $\beta_w(\varphi)$ (curve c), calculated from formulae (3.9). These ratios diverge at the critical angles $\varphi = \theta$ and $\varphi = -\pi + \theta$, to which the singularities in the roots of the dispersion equation correspond (as also in the asymptotic theory of the reflection of beams of internal waves [11, 13]). In addition $\beta_b(\varphi)$ has a singularity at $\varphi = -\theta$. The exact solutions are analytic everywhere.

The wave field for a beam propagating in the first quadrant, calculated from the force-source model, agrees with the exact solution only when the radiating surface is horizontal or is inclined at an angle $\varphi_0(\theta)$, which satisfied the equation $\beta_w(\varphi_0) = 1$. It is difficult to obtain an analytic expression for φ_0 (it is necessary to solve a cubic equation of general form). In the region $\varphi_0 < \varphi < 0$ the model solution is less accurate. In two special cases, to which the zeros of the function $\beta_w(\varphi)$ ($\varphi = -\theta$ and $\varphi = \theta - \pi/2$) correspond, this force source does not excite a beam of waves in the first quadrant, although it exists in the exact solution. When $\varphi = -\theta$ a comparison of the amplitudes makes no sense since the spatial structure of the beams, described by (2.8) and (2.10), becomes different.

The boundary flows in the force-source model are calculated incorrectly in all cases. The ratio $\beta_b(\varphi)$ diverges for three values of φ and exceeds unity for all the remaining values of φ. This result indicates a limitation on the possibility of employing the widely used force-source model to calculate both the wave drag and the total drag of bodies moving in a viscous inhomogeneous fluid.

4. THE SPATIAL STRUCTURE OF THE RADIATED BEAM

We will consider in more detail the special case when the waves are radiated by part of the plane in the form of a strip of width a, which performs vibrations of amplitude b. In a similar formulation in the experiment a beam of internal waves is excited by a thin rigid plate (the width a of which is much less than its length), which vibrates along its surface. In this case the spatial velocity spectrum $V(k)$ is described by (2.7), and the vertical displacements h of the particles, found from the solution of (2.8), have the form

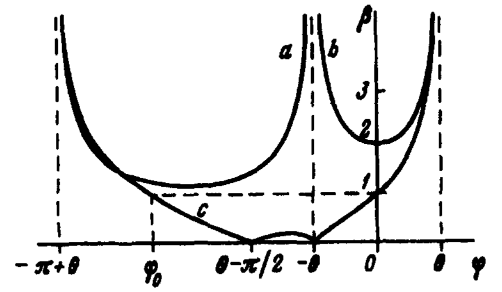

Fig. 2.

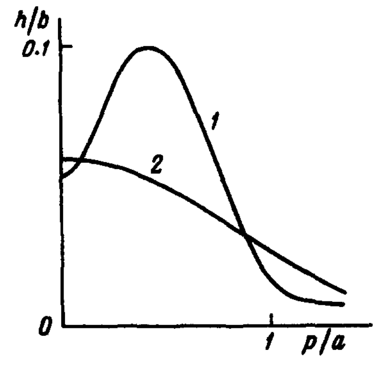

Fig. 3.

$$h = -(1+i\mu)\frac{\alpha b l_\nu \sin\theta}{6\sqrt{2\pi}}\left|\frac{\sin(\theta-\varphi)}{\sin(\theta+\varphi)}\right|^{1/2}\Phi(p,q)$$

$$\Phi(p,q) = F\left(p+\frac{a'}{2},q\right) - F\left(p-\frac{a'}{2},q\right)$$

$$F(p,q) = \int_0^\infty y^{-2/3}\exp(i\alpha p y^{1/3})e^{-y}dy, \quad \alpha = \frac{2N\cos\theta}{(\nu q)^{1/3}}, \quad a' = a\sin(\theta-\varphi)$$

where a' is the projection of the width of the plate onto the p axis, perpendicular to the beam axis.

We can represent the function $F(p, q)$ in the form of an expansion

$$F(p,q) = \sum_{n=0}^{\infty}\frac{(i\alpha p)^n}{n!}\Gamma\left(\frac{n+1}{3}\right) \tag{4.1}$$

having an infinite convergence radius with respect to αp.

In Fig. 3 we show, in relative units (the displacement h is normalized to the vibration amplitude of the plate b), the envelopes of the beam for two distances q from a plate of width a = 3 cm, situated at an angle of 30° to the horizontal and vibrating with a relative frequency $\omega/N = \sqrt{2}/2$ (in view of the symmetry of the beam we only show the region $p \geqslant 0$). It can be seen that at short distances (q/a = 4/3, curve 1) the beam has a bimodal structure, while at large distances (q/a = 40/3, curve 2) it is unimodal with a maximum at the centre.

The change from a bimodal structure to a unimodal structure occurs at distances L at which the relation $\partial^2|\Phi(p, L)|/\partial p^2|_{p=0} = 0$ is satisfied. Using expression (4.1) for $F(p, q)$, we obtain

$$L = a^3 N\cos\theta\sin^3(\theta-\varphi)/(4y^3\nu) \tag{4.2}$$

where y is the root of the equation

$$S^2(y,3) - S(y,2)S(y,4) = 0; \quad S(y,n) = \sum_{m=0}^{\infty}\frac{(-1)^m y^m}{(2m+1)!}\Gamma\left(\frac{2m+n}{3}\right)$$

Numerical solution of this equation showed that it has three real roots: y_1 = 8.148, y_2 = 7.824 and y_3 = 3.026, where, when $y_3 < y < y_1$, the derivative $\partial^2|\Phi(p, q)|/\partial p^2|_{p=0} = 0$ is small. Hence, two critical distances exist

$$L_1 \approx a^3 N\cos\theta\sin^3(\theta-\varphi)/(2000\nu), \quad L_2 \approx 20L_1$$

where, when $L < L_1$ the beam structure is bimodal, when $L_1 < L < L_2$ the bimodal structure changes into a unimodal structure and, finally, when $L > L_2$ the beam is unimodal.

Both in natural systems and in laboratory setups the stratification is weak and its scale $\Lambda = (d \ln \rho_0(z)/dz^{-1}$ (the distance at which fluid density changes by a factor of e) considerably exceeds all the other characteristic dimensions of the problem. Substituting $L = \Lambda$ into (4.2) we obtain that, for plate dimensions exceeding the viscous wave scale

$$L_\nu = (\nu\Lambda/N)^{1/3} = (\nu g)^{1/3}/N$$

the spatial structure of the beam will be bimodal over the whole space that is attainable in practice, which is also observed experimentally [7].

5. CONCLUSION

We have constructed an exact solution of the linearized problem of the excitation of the motions of a viscous homogeneous and stratified fluid by part of an inclined plane, which vibrates along its surface.

In the case of a vibrating plate, the correct procedure for calculating the internal-wave field is to find the spectral densities $B_j(k)$ and the stream function for a homogeneous fluid, replacing the wave numbers $k_1(k)$ and $k_2(k)$ by the wave numbers $\varkappa_1(k)$ and $\varkappa_2(k)$, which are solutions of the dispersion equation of the internal waves (2.2), and calculating the stream function from formula (2.1).

It follows from the analysis that, in the general case, the formal replacement of the actual body, radiating internal waves in an inhomogeneous fluid, by a set of force sources, obtained from the solution of the corresponding problem for a homogeneous fluid, leads to incorrect results. To calculate the internal-wave field, formed when a body moves in a stratified fluid, one must solve the equivalent problem for a homogeneous fluid, after which in momentum space (the space of wave numbers) one replaces the wave numbers (2.6) by the corresponding solutions of dispersion equation (2.2). The solutions constructed are analytic functions for all values of the physical parameters of the problem. The proposed method also enables three-dimensional problems to be considered.

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